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Darboux transformation and exact solutions for a hierarchy of nonlinear evolution equations

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Abstract
Starting from a spectral problem, we derive a hierarchy of nonlinear evolution equations. An explicit and universal Darboux transformation for the whole hierarchy is constructed. The exact solutions for the hierarchy are obtained by applying the Darboux transformation.

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1. Introduction

The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solutions, if available, of those nonlinear equations facilitate the verification of numerical solvers and aids in the stability analysis of solutions. In the past few decades, there has been significant progress in the development of various methods. Among them, the Darboux transformation is a powerful method to get exact solutions of nonlinear partial differential equations. The key for constructing Darboux transformation is to expose kinds of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations, including the KdV equation, KP equation, Davey–Stewartson equation, Yang–Mills flows, etc [1–8].

In this paper, we are interested in the Darboux transformation and exact solutions of a equation hierarchy associated with the following spectral problem:

$$\psi_x = U\psi = \begin{pmatrix} \lambda q & \lambda^2 + \lambda r + \frac{1}{2}(q^2 + r^2) \\ -\lambda^2 + \lambda r - \frac{1}{2}(q^2 + r^2) & -\lambda q \end{pmatrix} \psi, \qquad (1.1)$$

which was first introduced by Cao and Geng in [9], where they considered a particular case of the above spectral problem and showed that the associated involutive systems were generated from a confocal generator. Yan considered a more general case of the above spectral problem and showed that the eigenvalue problem was nonlinearized as a finite-dimensional completely integrable Hamiltonian system under the Bargmann constraint between the potentials and the eigenvalue functions [10].

The outline of our present paper is as follows. In section 2, we derive the equation hierarchy associated with the spectral problem (1.1). In section 3, we construct Darboux transformation for the hierarchy. In section 4, we construct exact solutions for the equation hierarchy by using its Darboux transformation.

2. The soliton hierarchy

In order to derive the isospectral hierarchy associated with (1.1), we consider the stationary zero-curvature equation

$$V_x = [U, V], \qquad V = \begin{pmatrix} a & b+c \\ b-c & -a \end{pmatrix} \tag{2.1}$$

with

$$a = \sum_{j\geqslant 0} a_{2j+1}\lambda^{-2j-1}, \qquad b = \sum_{j\geqslant 0} b_{2j+1}\lambda^{-2j-1}, \qquad c = \sum_{j\geqslant 0} c_{2j}\lambda^{-2j},$$

which gives rise to the following recurrence relations:

$$\begin{aligned} a_{2j+1} &= -\tfrac{1}{2}b_{2j-1x} + qc_{2j} - \tfrac{1}{2}(q^2+r^2)a_{2j-1}, \\ b_{2j+1} &= \tfrac{1}{2}a_{2j-1x} + rc_{2j} - \tfrac{1}{2}(q^2+r^2)b_{2j-1}, \\ c_{2jx} &= 2qb_{2j+1} - 2ra_{2j+1}. \end{aligned} \tag{2.2}$$

Then from (2.2), we further obtain the following recursive formula:

$$\begin{pmatrix} a_{2j+1} \\ b_{2j+1} \end{pmatrix} = L \begin{pmatrix} a_{2j-1} \\ b_{2j-1} \end{pmatrix}, \qquad j = 1, 2, \ldots, \tag{2.3}$$

where

$$L = \begin{pmatrix} q\partial^{-1}q\partial + q\partial^{-1}r(q^2+r^2) - \tfrac{1}{2}(q^2+r^2) & -q\partial^{-1}q(q^2+r^2) - \tfrac{1}{2}\partial + q\partial^{-1}r\partial \\ r\partial^{-1}r(q^2+r^2) + \tfrac{1}{2}\partial + r\partial^{-1}q\partial & r\partial^{-1}r\partial - r\partial^{-1}q(q^2+r^2) - \tfrac{1}{2}(q^2+r^2) \end{pmatrix}.$$

In order to derive the isospectral hierarchy associated with (1.1), we consider the auxiliary problem

$$\psi_{t_n} = V^{(n)}\psi = \left(\sum_{j=0}^{n}(M_{2j+1}\lambda^{2(n-j)+1} + N_{2j}\lambda^{2(n-j)+2}) + N_{2n+2}\right)\psi, \tag{2.4}$$

where

$$M_{2j+1} = \begin{pmatrix} a_{2j+1} & b_{2j+1} \\ b_{2j+1} & -a_{2j+1} \end{pmatrix}, \qquad N_{2j} = \begin{pmatrix} 0 & c_{2j} \\ -c_{2j} & 0 \end{pmatrix}.$$

The compatibility condition between (1.1) and (2.4) yields the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{2.5}$$

which is equivalent to the following hierarchy:

$$\begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = \begin{pmatrix} a_{2n+1x} - (q^2+r^2)b_{2n+1} + 2rc_{2n+2} \\ b_{2n+1x} + (q^2+r^2)b_{2n+1} - 2qc_{2n+2} \end{pmatrix}, \qquad n \geqslant 0. \tag{2.6}$$

Further we choose $a_1 = q, b_1 = r, c_0 = 1$. From (2.3) we can easily prove that $a_{2j+1}|_{(q,r)=(0,0)} = b_{2j+1}|_{(q,r)=(0,0)} = 0$ $(1 \leqslant j \leqslant n)$. We also need to use the condition $c_{2j}|_{(q,r)=(0,0)} = 0$ $(1 \leqslant j \leqslant n+1)$ to select the integration constant to be zero. Then $a_{2j+1}, b_{2j+1} (1 \leqslant j \leqslant n), c_{2j} (1 \leqslant j \leqslant n+1)$ can be uniquely determined by (2.3).

A direct calculation gives

$$\begin{aligned}
&c_2 = \tfrac{1}{2}(q^2+r^2), \qquad a_3 = -\tfrac{1}{2}r_x, \qquad b_3 = \tfrac{1}{2}q_x, \\
&c_4 = \tfrac{1}{2}(rq_x - qr_x) - \tfrac{1}{8}(q^2+r^2)^2, \\
&a_5 = -\tfrac{1}{4}q_{xx} - \tfrac{1}{2}q(qr_x - rq_x) - \tfrac{1}{8}q(q^2+r^2)^2 + \tfrac{1}{4}r_x(q^2+r^2), \\
&b_5 = -\tfrac{1}{4}r_{xx} - \tfrac{1}{2}r(qr_x - rq_x) - \tfrac{1}{8}r(q^2+r^2)^2 - \tfrac{1}{4}q_x(q^2+r^2), \\
&\cdots .
\end{aligned} \tag{2.7}$$

Typical nonlinear systems $(n = 1)$ in the hierarchy are

$$\begin{aligned}
q_{t_1} &= -\tfrac{1}{2}r_{xx} - r(qr_x - rq_x) - \tfrac{1}{4}r(q^2+r^2)^2 - \tfrac{1}{2}q_x(q^2+r^2), \\
r_{t_1} &= \tfrac{1}{2}q_{xx} + q(qr_x - rq_x) + \tfrac{1}{4}q(q^2+r^2)^2 - \tfrac{1}{2}r_x(q^2+r^2),
\end{aligned} \tag{2.8}$$

which are a new system of generalized derivative nonlinear Schrödinger equations (GDNS).

If we take $q = \mathrm{Im}(\psi), r = \mathrm{Re}(\psi)$, then (2.8) reduces to

$$\mathrm{i}\psi_t = \tfrac{1}{2}\psi_{xx} - \tfrac{1}{2}\mathrm{i}|\psi|^2\psi_x + \psi\,\mathrm{Im}(\psi\psi_x^*) + \tfrac{1}{4}\psi|\psi|^4,$$

which is the general form of the derivative nonlinear Schrödinger equation [11].

3. Darboux transformation

In this section, we will construct a Darboux transformation for the hierarchy (2.6). The Darboux transformation is actually a special gauge transformation

$$\widetilde{\psi} = T\psi \tag{3.1}$$

of the Lax pairs (1.1) and (2.4). It is required that $\widetilde{\psi}$ also satisfies Lax pairs (1.1) and (2.4) with some $\widetilde{U}$ and $\widetilde{V}^{(n)}$, i.e.

$$\widetilde{\psi}_x = \widetilde{U}\widetilde{\psi}, \qquad \widetilde{U} = (T_x + TU)T^{-1}, \tag{3.2}$$

$$\widetilde{\psi}_t = \widetilde{V}^{(n)}\widetilde{\psi}, \qquad \widetilde{V}^{(n)} = (T_t + TV^{(n)})T^{-1}. \tag{3.3}$$

By cross differentiating (3.2) and (3.3), we get

$$\widetilde{U}_t - \widetilde{V}_x^{(n)} + [\widetilde{U}, \widetilde{V}^{(n)}] = T\big(U_t - V_x^{(n)} + [U, V^{(n)}]\big)T^{-1}, \tag{3.4}$$

which imply that in order to make system (2.6) invariant under the gauge transformation (3.1), we should require $\widetilde{U}, \widetilde{V}^{(n)}$ have the same forms as $U, V^{(n)}$ respectively. At the same time the old potentials q, r in $U, V^{(n)}$ will be mapped into new potentials $\widetilde{q}, \widetilde{r}$ in $\widetilde{U}, \widetilde{V}^{(n)}$. This process can be iterated, and usually it yields a sequence of exact solutions for system (2.6). Following the idea of [2], we can construct the Darboux transformation for the equation hierarchy (2.6).

We define that

$$U_0 = \begin{pmatrix} 0 & \frac{1}{2}(q^2+r^2) \\ -\frac{1}{2}(q^2+r^2) & 0 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} q & r \\ r & -q \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then U can written as

$$U = U_0 + U_1\lambda + U_2\lambda^2. \tag{3.5}$$

Let $h = (h_1, h_2)^T$ be a solution of spectral problem (1.1) and (2.4) when $\lambda = \lambda_0$ ($\lambda_0 \neq 0$); then it is easy to see that $h^- = (h_2, -h_1)^T$ is also a solution of the spectral problem (1.1) and (2.4) when $\lambda = -\lambda_0$. We construct a new matrix

$$H = (h, h^-).$$

By using (1.1) and (2.4), we can get that

$$\begin{aligned} H_x &= U_0 H + U_1 H\Lambda + U_2 H\Lambda^2, \\ H_{t_n} &= \sum_{j=0}^{n} (M_{2j+1} H\Lambda^{2(n-j)+1} + N_{2j} H\Lambda^{2(n-j)+2}) + N_{2n+2} H, \end{aligned} \tag{3.6}$$

where

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}.$$

We construct the Darboux matrix

$$T = \lambda I + S, \tag{3.7}$$

where

$$\begin{aligned} S &= -H\Lambda H^{-1} \\ &= \frac{1}{h_1^2 + h_2^2} \begin{pmatrix} -\lambda_0 (h_1^2 - h_2^2) & -2\lambda_0 h_1 h_2 \\ -2\lambda_0 h_1 h_2 & \lambda_0 (h_1^2 - h_2^2) \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}. \end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.2), we can get that

$$\widetilde{U} = \widetilde{U}_0 + \widetilde{U}_1 \lambda + \widetilde{U}_2 \lambda^2, \tag{3.9}$$

where $\widetilde{U}_0$, $\widetilde{U}_1$ and $\widetilde{U}_2$ are determined by the following equations:

$$\begin{aligned} &\widetilde{U}_2 = U_2, && \widetilde{U}_1 = U_1 + SU_2 - \widetilde{U}_2 S, \\ &\widetilde{U}_0 = U_0 + SU_1 - \widetilde{U}_1 S, && \widetilde{U}_0 S = S_x + SU_0. \end{aligned} \tag{3.10}$$

With the help of the first equation of (3.6), we can prove that the third equation of (3.10) is equivalent to the fourth one if the first two equations of (3.10) hold.

Substituting (3.7) into (3.3), we can get that

$$\widetilde{V}^{(n)} = \sum_{j=0}^{n} (\widetilde{M}_{2j+1} \lambda^{2(n-j)+1} + \widetilde{N}_{2j} \lambda^{2(n-j)+2}) + \widetilde{N}_{2n+2}, \tag{3.11}$$

where $\widetilde{M}_{2j+1}$, $\widetilde{N}_{2j}$ and $\widetilde{N}_{2n+2}$ are determined by the following equations:

$$\begin{aligned} &\widetilde{N}_0 = N_0, \\ &\widetilde{M}_{2j+1} = M_{2j+1} + SN_{2j} - \widetilde{N}_{2j} S, \\ &\widetilde{N}_{2j+2} = N_{2j+2} + SM_{2j+1} - \widetilde{M}_{2j+1} S, \qquad 1 \leqslant j \leqslant n \\ &\widetilde{N}_{2n+2} = N_{2n+2} + SM_{2n+1} - \widetilde{M}_{2n+1} S. \end{aligned} \tag{3.12}$$

Next we will prove that $\widetilde{U}$ and $\widetilde{V}^{(n)}$ have the same forms as U and $V^{(n)}$ after some transformations, respectively.

Proposition 1. *The matrix* $\widetilde{U}$ *determined by (3.9) has the same form as* U*, that is,*

$$\widetilde{U} = \widetilde{U}_0 + \widetilde{U}_1\lambda + \widetilde{U}_2\lambda^2 = \begin{pmatrix} 0 & \frac{1}{2}(\widetilde{q}^2+\widetilde{r}^2) \\ -\frac{1}{2}(\widetilde{q}^2+\widetilde{r}^2) & 0 \end{pmatrix} + \begin{pmatrix} \widetilde{q} & \widetilde{r} \\ \widetilde{r} & -\widetilde{q} \end{pmatrix}\lambda + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\lambda^2, \tag{3.13}$$

where the transformations between q, r *and* $\widetilde{q}, \widetilde{r}$ *are given by*

$$\widetilde{q} = q - 2\beta, \qquad \widetilde{r} = r + 2\alpha, \tag{3.14}$$

where α, β *are determined by (3.8). The transformation* $(\psi, q, r) \to (\widetilde{\psi}, \widetilde{q}, \widetilde{r})$ *is called a Darboux transformation of the spectral problem (1.1).*

Proof. From (3.10), we obtain that

$$\widetilde{U}_2 = U_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\begin{aligned}\widetilde{U}_1 &= U_1 + SU_2 - \widetilde{U}_2 S \\ &= \begin{pmatrix} q & r \\ r & -q \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \\ &= \begin{pmatrix} q-2\beta & r+2\alpha \\ r+2\alpha & -q+2\beta \end{pmatrix} = \begin{pmatrix} \widetilde{q} & \widetilde{r} \\ \widetilde{r} & -\widetilde{q} \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\widetilde{U}_0 &= U_0 + SU_1 - \widetilde{U}_1 S \\ &= \begin{pmatrix} 0 & \frac{1}{2}(q^2+r^2) \\ -\frac{1}{2}(q^2+r^2) & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} q & r \\ r & -q \end{pmatrix} \\ &\quad - \begin{pmatrix} q-2\beta & r+2\alpha \\ r+2\alpha & -q+2\beta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2}(q^2+r^2)+2\alpha r-2\beta q+2\alpha^2+2\beta^2 \\ -\frac{1}{2}(q^2+r^2)-2\alpha r+2\beta q-2\alpha^2-2\beta^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2}[(q-2\beta)^2+(r+2\alpha)^2] \\ -\frac{1}{2}[(q-2\beta)^2+(r+2\alpha)^2] & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2}(\widetilde{q}^2+\widetilde{r}^2) \\ -\frac{1}{2}(\widetilde{q}^2+\widetilde{r}^2) & 0 \end{pmatrix}.\end{aligned}$$

So we get that the transformation (3.14) holds and $\widetilde{U}$ has the same form as U. The proof is completed. □

Next we will prove that $\widetilde{V}^{(n)}$ also has the same form as $V^{(n)}$ under the transformations (3.1) and (3.14).

Proposition 2. *The matrix* $\widetilde{V}^{(n)}$ *determined by (3.11) has the same form as* $V^{(n)}$ *under the transformations (3.1) and (3.14).*

Proof. Because $\widetilde{V}^{(n)}$ can be expressed as

$$\widetilde{V}^{(n)} = \sum_{j=0}^{n}(\widetilde{M}_{2j+1}\lambda^{2(n-j)+1} + \widetilde{N}_{2j}\lambda^{2(n-j)+2}) + N_{2n+2},$$

we only need to prove that $\widetilde{M}_{2j+1}, 0 \leqslant j \leqslant n$, and $\widetilde{N}_{2j}, 0 \leqslant j \leqslant n+1$, have the same forms as M_{2j+1} and N_{2j} under the transformation (3.1) and (3.14), respectively.

From (3.12), we get that

$$\widetilde{N}_0 = N_0 = \begin{pmatrix} 0 & c_0 \\ -c_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{c}_0 \\ -\widetilde{c}_0 & 0 \end{pmatrix}, \tag{3.15}$$

$$\begin{aligned} \widetilde{M}_1 &= M_1 + SN_0 - \widetilde{N}_0 S \\ &= \begin{pmatrix} q & r \\ r & -q \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \\ &= \begin{pmatrix} q - 2\beta & r + 2\alpha \\ r + 2\alpha & -q + 2\beta \end{pmatrix} = \begin{pmatrix} \widetilde{a}_1 & \widetilde{b}_1 \\ \widetilde{b}_1 & -\widetilde{a}_1 \end{pmatrix}, \end{aligned} \tag{3.16}$$

where

$$\widetilde{c}_0 = 1, \qquad \widetilde{a}_1 = \widetilde{q}, \qquad \widetilde{b}_1 = \widetilde{r}. \tag{3.17}$$

Thus, we get that $\widetilde{M}_1$ and $\widetilde{N}_0$, respectively, have the same forms as M_1 and N_0 after the transformations.

Again by using (3.12), (3.15) and (3.16), through some calculations, we obtain that $\widetilde{M}_{2j+1}$ $(1 \leqslant j \leqslant n)$ and $\widetilde{N}_{2j}$ $(1 \leqslant j \leqslant n+1)$ have the following forms:

$$\widetilde{M}_{2j+1} = \begin{pmatrix} \widetilde{a}_{2j+1} & \widetilde{b}_{2j+1} \\ \widetilde{b}_{2j+1} & -\widetilde{a}_{2j+1} \end{pmatrix}, \qquad \widetilde{N}_{2j} = \begin{pmatrix} 0 & \widetilde{c}_{2j} \\ -\widetilde{c}_{2j} & 0 \end{pmatrix}, \tag{3.18}$$

where

$$\begin{aligned} \widetilde{a}_{2j+1} &= a_{2j+1} - \beta(c_{2j} + \widetilde{c}_{2j}), \\ \widetilde{b}_{2j+1} &= b_{2j+1} + \alpha(c_{2j} + \widetilde{c}_{2j}), \\ \widetilde{c}_{2j} &= c_{2j} + \alpha(b_{2j-1} + \widetilde{b}_{2j-1}) - \beta(a_{2j-1} + \widetilde{a}_{2j-1}). \end{aligned} \tag{3.19}$$

Next we only need to prove that $\widetilde{a}_{2j+1}, \widetilde{b}_{2j+1}$ $(1 \leqslant j \leqslant n)$ and $\widetilde{c}_{2j} (1 \leqslant j \leqslant n+1)$ have the same forms as a_{2j+1}, b_{2j+1} and c_{2j}, respectively, after the transformation (3.14).

By using (2.2) and (3.4), we have

$$\widetilde{U}_t - \widetilde{V}^{(n)}_x + [\widetilde{U}, \widetilde{V}^{(n)}] = T\big(U_t - V^{(n)}_x + [U, V^{(n)}]\big)T^{-1} = 0, \tag{3.20}$$

which is equivalent to the following recurrence relations:

$$\begin{pmatrix} \widetilde{a}_{2j+1} \\ \widetilde{b}_{2j+1} \end{pmatrix} = \widetilde{L}\begin{pmatrix} \widetilde{a}_{2j-1} \\ \widetilde{b}_{2j-1} \end{pmatrix}, \qquad j = 1, 2, \ldots, \tag{3.21}$$

where

$$\widetilde{L} = \begin{pmatrix} \widetilde{q}\partial^{-1}\widetilde{q}\partial + \widetilde{q}\partial^{-1}r(\widetilde{q}^2 + \widetilde{r}^2) - \frac{1}{2}(\widetilde{q}^2 + \widetilde{r}^2) & -\widetilde{q}\partial^{-1}\widetilde{q}(\widetilde{q}^2 + \widetilde{r}^2) - \frac{1}{2}\partial + \widetilde{q}\partial^{-1}\widetilde{r}\partial \\ \widetilde{r}\partial^{-1}\widetilde{r}(\widetilde{q}^2 + \widetilde{r}^2) + \frac{1}{2}\partial + \widetilde{r}\partial^{-1}\widetilde{q}\partial & \widetilde{r}\partial^{-1}\widetilde{r}\partial - \widetilde{r}\partial^{-1}\widetilde{q}(\widetilde{q}^2 + \widetilde{r}^2) - \frac{1}{2}(\widetilde{q}^2 + \widetilde{r}^2) \end{pmatrix},$$

and the equations

$$\begin{pmatrix} \widetilde{q}_{t_n} \\ \widetilde{r}_{t_n} \end{pmatrix} = \begin{pmatrix} \widetilde{a}_{2n+1x} - (\widetilde{q}^2 + \widetilde{r}^2)\widetilde{b}_{2n+1} + 2\widetilde{r}\widetilde{c}_{2n+2} \\ \widetilde{b}_{2n+1x} + (\widetilde{q}^2 + \widetilde{r}^2)\widetilde{b}_{2n+1} - 2\widetilde{q}\widetilde{c}_{2n+2} \end{pmatrix}. \tag{3.22}$$

From (3.21), we can easily prove that

$$\widetilde{a}_{2j+1}|_{(\widetilde{q},\widetilde{r})=(0,0)} = \widetilde{b}_{2j+1}|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0 \ (1 \leqslant j \leqslant n).$$

Again by using (3.21), (3.14) and (3.19), we have

$$\widetilde{c}_{2jx}|_{(\widetilde{q},\widetilde{r})=(0,0)} = 2\widetilde{q}\widetilde{b}_{2j+1} - 2\widetilde{r}\widetilde{a}_{2j+1}|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0.$$

On the other hand,

$$\begin{aligned} \widetilde{c}_{2jx} &= \partial_x(c_{2j} + \alpha(b_{2j-1} + \widetilde{b}_{2j-1}) - \beta(a_{2j-1} + \widetilde{a}_{2j-1}))|_{(\widetilde{q},\widetilde{r})=(0,0)} \\ &= \partial_x(c_{2j} + \alpha b_{2j-1} - \beta a_{2j-1})|_{(\widetilde{q},\widetilde{r})=(0,0)}, \qquad 1 \leqslant j \leqslant n+1. \end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{c}_{2j}|_{(\widetilde{q},\widetilde{r})=(0,0)} &= c_{2j} + \alpha b_{2j-1} - \beta a_{2j-1}|_{(\widetilde{q},\widetilde{r})=(0,0)} \\ &= f(t).\end{aligned} \tag{3.23}$$

Note the fact

$$c_{2j}|_{(q,r)=(0,0)} = b_{2j-1}|_{(q,r)=(0,0)} = a_{2j-1}|_{(q,r)=(0,0)} = 0,$$

so the integral constant $f(t)$ must be zero, i.e.

$$\widetilde{c}_{2j}|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0, \qquad 1 \leqslant j \leqslant n+1.$$

We proved that

$$\widetilde{a}_{2j+1}, \widetilde{b}_{2j+1}\ (1 \leqslant j \leqslant n), \qquad \widetilde{c}_{2j}\ (1 \leqslant j \leqslant n+1)$$

satisfy the same equations and the same boundary conditions with $a_{2j+1}, b_{2j+1}, c_{2j}$, so they must have the same forms. The proof is completed. □

From propositions 1 and 2, we get the following theorem.

Theorem 1. *The solutions (q, r) of the hierarchy (2.6) are mapped into their new solutions $(\widetilde{q}, \widetilde{r})$ under the Darboux transformations (3.1) and (3.14), where α, β are given by (3.8).*

4. Applications of Darboux transformations

In this section, we will apply the Darboux transformation (3.14) to construct explicit solutions of the hierarchy (2.6). As usual we make the Darboux transformation starting from a special solution of (2.6). We start from $q = q_0$, $r = r_0$, and we choose

$$h^{(k)} = \begin{pmatrix} h_1^{(k)} \\ h_2^{(k)} \end{pmatrix}, \qquad 1 \leqslant k \leqslant N, \tag{4.1}$$

as the solutions of the Lax pairs (1.1) and (2.4) when $\lambda = \lambda_k$. Then we could construct a series of exact solutions of (2.6) as follows.

First, we construct

$$\begin{aligned} H^{(1)} &= (h^{(1)}, h^{(1)-}), \qquad \Lambda^{(1)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \\ S^{(1)} &= -H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1} \\ &= \frac{1}{h_1^{(1)2} + h_2^{(1)2}} \begin{pmatrix} -\lambda_1\left(h_1^{(1)2} - h_2^{(1)2}\right) & -2\lambda_1 h_1^{(1)} h_2^{(1)} \\ -2\lambda_1 h_1^{(1)} h_2^{(1)} & \lambda_1\left(h_1^{(1)2} - h_2^{(1)2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{(1)} & \beta^{(1)} \\ \beta^{(1)} & -\alpha^{(1)} \end{pmatrix}. \end{aligned} \tag{4.2}$$

Then by the use of theorem.1, we can get the new solutions $q[1], r[1]$ of (2.6) from the following equations:

$$q[1] = q_0 - 2\beta^{(1)}, \qquad r[1] = r_0 + 2\alpha^{(1)}. \tag{4.3}$$

With the help of (3.1), (3.7) and after some calculations, we can get the solutions of Lax pairs (1.1) and (2.4), when $q = q[1]$, $r = r[1]$ and $\lambda = \lambda_i$. These solutions can be expressed as

$$\overline{h}_{[1]}^{(i)} = \begin{pmatrix} \overline{h}_{1[1]}^{(i)} \\ \overline{h}_{2[1]}^{(i)} \end{pmatrix} = (\lambda_i I + S^{(1)}) \begin{pmatrix} h_1^{(i)} \\ h_2^{(i)} \end{pmatrix} = \frac{\lambda_1 + \lambda_i}{[h^{(1)}, h^{(1)}]} \begin{pmatrix} \begin{vmatrix} h_1^{(i)} & [h^{(i)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix} \\ \begin{vmatrix} h_2^{(i)} & [h^{(i)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix} \end{pmatrix}, \tag{4.4}$$

where

$$[h^{(i)}, h^{(j)}] = \frac{h_1^{(i)} h_1^{(j)} + h_2^{(i)} h_2^{(j)}}{\lambda_i + \lambda_j}.$$

We construct

$$\begin{aligned}
H^{(2)} &= \left(\overline{h}^{(2)}_{[1]}, \overline{h}^{(2)-}_{[1]}\right), \qquad \Lambda^{(2)} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \\
S^{(2)} &= -H^{(2)} \Lambda^{(2)} (H^{(2)})^{-1} \\
&= \frac{1}{\overline{h}^{(2)2}_{1[1]} + \overline{h}^{(2)2}_{2[1]}} \begin{pmatrix} -\lambda_2\left(\overline{h}^{(2)2}_{1[1]} - \overline{h}^{(2)2}_{2[1]}\right) & -2\lambda_2 \overline{h}^{(2)}_{1[1]} \overline{h}^{(2)}_{2[1]} \\ -2\lambda_2 \overline{h}^{(2)}_{1[1]} \overline{h}^{(2)}_{2[1]} & \lambda_2\left(\overline{h}^{(\overline{1})2}_{1[1]} - \overline{h}^{(2)2}_{2[1]}\right) \end{pmatrix} \\
&= \begin{pmatrix} \alpha^{(2)} & \beta^{(2)} \\ \beta^{(2)} & -\alpha^{(2)} \end{pmatrix}.
\end{aligned} \tag{4.5}$$

Then we can get the new solutions $q[2], r[2]$ of (2.6) from the following equations:

$$q[2] = q[1] - 2\beta^{(2)}, \qquad r[2] = r[1] + 2\alpha^{(2)}. \tag{4.6}$$

By a direct calculation, we have the following formulae:

$$\overline{h}^{(2)2}_{1[1]} + \overline{h}^{(2)2}_{2[1]} = \frac{2\lambda_2(\lambda_1 + \lambda_2)^2}{[h^1, h^1]^2}[h^1, h^1]([h^1, h^1][h^2, h^2] - [h^2, h^1]^2), \tag{4.7a}$$

$$\begin{aligned}
\overline{h}^{(2)}_{1[1]} \overline{h}^{(2)}_{2[1]} = \frac{(\lambda_1 + \lambda_2)^2}{[h^1, h^1]^2}\big([h^1, h^1]^2 h_1^{(2)} h_2^{(2)} + [h^2, h^1]^2 h_1^{(1)} h_2^{(1)} \\
- [h^1, h^1][h^2, h^1]\big(h_1^{(1)} h_2^{(2)} + h_1^{(2)} h_2^{(1)}\big)\big),
\end{aligned} \tag{4.7b}$$

$$\overline{h}^{(2)2}_{1[1]} = \frac{(\lambda_1 + \lambda_2)^2}{[h^1, h^1]^2}\big([h^1, h^1]^2 h_1^{(2)2} + [h^2, h^1]^2 h_1^{(1)2} - 2[h^1, h^1][h^2, h^1] h_1^{(1)} h_1^{(2)}\big). \tag{4.7c}$$

So with the help of (4.4), (4.5), (4.7), after taking the Darboux transformation once again, we can obtain the solution of Lax pairs (1.1) and (2.4) when $q = q[2], r = r[2], \lambda = \lambda_3$,

$$\begin{aligned}
\overline{h}^{(3)}_{[2]} &= \begin{pmatrix} \overline{h}^{(3)}_{1[2]} \\ \overline{h}^{(3)}_{2[2]} \end{pmatrix} = (\lambda_3 I + S^{(2)}) \begin{pmatrix} \overline{h}^{(3)}_{1[1]} \\ \overline{h}^{(3)}_{2[1]} \end{pmatrix} \\
&= \frac{(\lambda_3 + \lambda_2)(\lambda_3 + \lambda_1)}{[h^1, h^1][h^2, h^2] - [h^2, h^1]^2} \begin{pmatrix} \begin{vmatrix} h_1^{(3)} & [h^{(3)}, h^{(1)}] & [h^{(3)}, h^{(2)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] & [h^{(1)}, h^{(2)}] \\ h_1^{(2)} & [h^{(2)}, h^{(1)}] & [h^{(2)}, h^{(2)}] \end{vmatrix} \\ \begin{vmatrix} h_2^{(3)} & [h^{(3)}, h^{(1)}] & [h^{(3)}, h^{(2)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] & [h^{(1)}, h^{(2)}] \\ h_2^{(2)} & [h^{(2)}, h^{(1)}] & [h^{(2)}, h^{(2)}] \end{vmatrix} \end{pmatrix}.
\end{aligned} \tag{4.8}$$

If we have done the Darboux transformation $N - 1$ times and got the solutions of (2.6) as $q[N-1], r[N-1]$, we can express the solutions of Lax pairs (1.1) and (2.4)

$(q = q[N-1], r = r[N-1], \lambda = \lambda_N)$ as follows:

$$\overline{h}^{(N)}_{[N-1]} = \begin{pmatrix} \overline{h}^{(N)}_{1[N-1]} \\ \overline{h}^{(N)}_{2[N-1]} \end{pmatrix} = \Delta_N \begin{pmatrix} \begin{vmatrix} h_1^{(N)} & [h^{(N)}, h^{(1)}] & \cdots & [h^{(N)}, h^{(N-1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] & \cdots & [h^{(1)}, h^{(N-1)}] \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(N-1)} & [h^{(N-1)}, h^{(1)}] & \cdots & [h^{(N-1)}, h^{(N-1)}] \end{vmatrix} \\ \begin{vmatrix} h_2^{(N)} & [h^{(N)}, h^{(1)}] & \cdots & [h^{(N)}, h^{(N-1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] & \cdots & [h^{(1)}, h^{(N-1)}] \\ \vdots & \vdots & \ddots & \vdots \\ h_2^{(N-1)} & [h^{(N-1)}, h^{(1)}] & \cdots & [h^{(N-1)}, h^{(N-1)}] \end{vmatrix} \end{pmatrix}, \tag{4.9}$$

where

$$\Delta_N = \frac{(\lambda_N + \lambda_1)(\lambda_N + \lambda_2)\cdots(\lambda_N + \lambda_{N-1})}{\begin{vmatrix} [h^{(1)}, h^{(1)}] & [h^{(1)}, h^{(2)}] & \cdots & [h^{(1)}, h^{(N-1)}] \\ [h^{(2)}, h^{(1)}] & [h^{(2)}, h^{(2)}] & \cdots & [h^{(2)}, h^{(N-1)}] \\ \vdots & \vdots & \ddots & \vdots \\ [h^{(N-1)}, h^{(1)}] & [h^{(N-1)}, h^{(2)}] & \cdots & [h^{(N-1)}, h^{(N-1)}] \end{vmatrix}}.$$

We construct

$$H^{(N)} = \left(\overline{h}^{(N)}_{[N-1]}, \overline{h}^{(N)-}_{[N-1]}\right), \qquad \Lambda^{(N)} = \begin{pmatrix} \lambda_N & 0 \\ 0 & -\lambda_N \end{pmatrix},$$

$$\begin{aligned} S^{(N)} &= -H^{(N)}\Lambda^{(N)}(H^{(N)})^{-1} \\ &= \frac{1}{\overline{h}^{(2)2}_{1[N-1]} + \overline{h}^{(2)2}_{2[N-1]}} \begin{pmatrix} -\lambda_N\left(\overline{h}^{(2)2}_{1[N-1]} - \overline{h}^{(2)2}_{2[N-1]}\right) & -2\lambda_N \overline{h}^{(2)}_{1[N-1]}\overline{h}^{(2)}_{2[N-1]} \\ -2\lambda_N \overline{h}^{(2)}_{1[N-1]}\overline{h}^{(2)}_{2[N-1]} & \lambda_N\left(\overline{h}^{(\overline{1})2}_{1[N-1]} - \overline{h}^{(2)2}_{2[N-1]}\right) \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{(N)} & \beta^{(N)} \\ \beta^{(N)} & -\alpha^{(N)} \end{pmatrix}. \end{aligned} \tag{4.10}$$

Then we can get the new solutions $q[N], r[N]$ of (2.6) from the following equations:

$$q[N] = q[N-1] - 2\beta^{(N)}, \qquad r[N] = r[N-1] + 2\alpha^{(N)}. \tag{4.11}$$

This process can be iterated and usually it yields a sequence of new solutions.

In the end, we will give a simple example. We will construct the exact solutions for the hierarchy (2.6). Substituting $q = 0, r = 0$ into the Lax pairs (1.1) and (2.4), we choose basic solutions corresponding to $\lambda = \lambda_1 = r_1 e^{\frac{\pi}{4}i}$, i.e. $\lambda_1^2 = \mathrm{i} r_1^2$ as follows:

$$h^{(1)} = \begin{pmatrix} c_1 e^{\theta_1} + c_2 e^{-\theta_1} \\ -\mathrm{i}(c_1 e^{\theta_1} - c_2 e^{-\theta_1}) \end{pmatrix}, \tag{4.12}$$

where $\theta_1 = r_1^2 x + r_1^{2n+2} t$, $n \in 4\mathbf{Z}^+$, and c_1, c_2 are non-zero constants. We construct

$$H^{(1)} = (h^{(1)}, h^{(1)-}) = \begin{pmatrix} c_1 e^{\theta_1} + c_2 e^{-\theta_1} & -\mathrm{i}(c_1 e^{\theta_1} - c_2 e^{-\theta_1}) \\ -\mathrm{i}(c_1 e^{\theta_1} - c_2 e^{-\theta_1}) & -(c_1 e^{\theta_1} + c_2 e^{-\theta_1}) \end{pmatrix}, \quad \Lambda^{(1)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix},$$

$$\begin{aligned} S^{(1)} &= -H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1} = \begin{pmatrix} \alpha^{(1)} & \beta^{(1)} \\ \beta^{(1)} & -\alpha^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1/2\lambda_1\left(c_1^2 e^{2\theta_1} + c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} & \frac{1/2\mathrm{i}\lambda_1\left(c_1^2 e^{2\theta_1} - c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} \\ \frac{1/2\mathrm{i}\lambda_1\left(c_1^2 e^{2\theta_1} - c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} & \frac{1/2\lambda_1\left({c_1}^2 e^{2\theta_1} + c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} \end{pmatrix}. \end{aligned} \tag{4.13}$$

Thus from (3.14), we can get

$$
\begin{aligned}
q[1] &= -2\beta^{(1)} = -\frac{\mathrm{i}\lambda_1\left(c_1^2 e^{2\theta_1} - c_2^2 e^{-2\theta_1}\right)}{c_1 c_2}, \\
r[1] &= 2\alpha^{(1)} = -\frac{\lambda_1\left(c_1^2 e^{2\theta_1} + c_2^2 e^{-2\theta_1}\right)}{c_1 c_2}.
\end{aligned} \tag{4.14}
$$

If we choose another basic solution corresponding to $\lambda = \lambda_2 = r_2 e^{\frac{\pi}{4}\mathrm{i}}$, $r_2 \neq r_1$ as follows:

$$
h^{(2)} = \begin{pmatrix} c_3 e^{\theta_2} - c_4 e^{-\theta_2} \\ -\mathrm{i}(c_3 e^{\theta_1} + c_4 e^{-\theta_2}) \end{pmatrix}, \tag{4.15}
$$

where $\theta_2 = r_2^2 x + r_2^{2n+2} t$, $n \in 4\mathbf{Z}^+$, and c_3, c_4 are non-zero constants.

From (4.4), we have

$$
\overline{h}^{(2)}_{[1]} = \begin{pmatrix} \overline{h}^{(2)}_{1[1]} \\ \overline{h}^{(2)}_{2[1]} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2 c_1 c_2 c_3 e^{\theta_2} - \lambda_2 c_1 c_2 c_4 e^{-\theta_2} + \lambda_1 c_1^2 c_4 e^{2\theta_1 - \theta_2} - \lambda_1 c_2^2 c_3 e^{-2\theta_1 + \theta_2}}{c_1 c_2} \\ \frac{-\mathrm{i}\left(\lambda_1 {c_1}^2 c_4 e^{2\theta_1 - \theta_2} + \lambda_1 {c_2}^2 c_3 e^{-2\theta_1 + \theta_2} + \lambda_2 c_1 c_2 c_3 e^{\theta_2} + \lambda_2 c_1 c_2 c_4 e^{-\theta_2}\right)}{c_1 c_2} \end{pmatrix}.
$$

We construct

$$
H^{(2)} = \left(\overline{h}^{(2)}_{[1]}, \overline{h}^{(2)-}_{[1]}\right), \qquad \Lambda^{(2)} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix},
$$

$$
\begin{aligned}
S^{(2)} &= -H^{(2)}\Lambda^{(2)}(H^{(2)})^{-1} \\
&= \frac{1}{\overline{h}^{(2)2}_{1[1]} + \overline{h}^{(2)2}_{2[1]}} \begin{pmatrix} -\lambda_2\left(\overline{h}^{(2)2}_{1[1]} - \overline{h}^{(2)2}_{2[1]}\right) & -2\lambda_2 \overline{h}^{(2)}_{1[1]}\overline{h}^{(2)}_{2[1]} \\ -2\lambda_2 \overline{h}^{(2)}_{1[1]}\overline{h}^{(2)}_{2[1]} & \lambda_2\left(\overline{h}^{(\bar{1})2}_{1[1]} - \overline{h}^{(2)2}_{2[1]}\right) \end{pmatrix} \\
&= \begin{pmatrix} \alpha^{(2)} & \beta^{(2)} \\ \beta^{(2)} & -\alpha^{(2)} \end{pmatrix}.
\end{aligned}
$$

With the help of Maple, we can get another exact solution of system (2.6) from (4.8)

$$
\begin{aligned}
q[2] &= -\frac{\mathrm{i}\lambda_1\left(c_1^2 e^{2\theta_1} - c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} - 2\beta^{(2)}, \\
r[2] &= -\frac{\lambda_1\left(c_1^2 e^{2\theta_1} + c_2^2 e^{-2\theta_1}\right)}{c_1 c_2} + 2\alpha^{(2)},
\end{aligned} \tag{4.16}
$$

where

$$
\begin{aligned}
\alpha^{(2)} = {} & \big[1/2\lambda_2\big(\lambda_2^2 c_1^2 c_2^2 {c_3}^2 e^{2\theta_2} + 2\lambda_2\lambda_1 {c_1}^3 c_2 c_3 c_4 e^{2\theta_1} + {\lambda_2}^2 c_1^2 c_2^2 {c_4}^2 e^{-2\theta_2} \\
& + 2\lambda_2\lambda_1 c_1 {c_2}^3 c_3 c_4 e^{-2\theta_1} + {\lambda_1}^2 {c_1}^4 c_4^2 e^{4\theta_1 - 2\theta_2} + {\lambda_1}^2 {c_2}^4 c_3^2 e^{-4\theta_1 + 2\theta_2}\big)\big] \\
& \times \big[c_1 c_2\big({\lambda_2}^2 c_1 c_2 c_3 c_4 + \lambda_2\lambda_1 {c_2}^2 {c_3}^2 e^{2\theta_2 - 2\theta_1} + \lambda_2\lambda_1 c_1^2 c_4^2 e^{-2\theta_2 + 2\theta_1} + {\lambda_1}^2 c_1 c_2 c_3 c_4\big)\big]^{-1} \\
\beta^{(2)} = {} & \big[-1/2\mathrm{i}\lambda_2\left(\lambda_2 c_1 c_2 c_3 e^{\theta_2} - \lambda_2 c_1 c_2 c_4 e^{-\theta_2} + \lambda_1 c_1^2 c_4 e^{2\theta_1 - \theta_2} - \lambda_1 c_2^2 c_3 e^{-2\theta_1 + \theta_2}\right) \\
& \times \left(\lambda_1 c_1^2 c_4 e^{2\theta_1 - \theta_2} + \lambda_1 c_2^2 c_3 e^{-2\theta_1 + \theta_2} + \lambda_2 c_1 c_2 c_3 e^{\theta_2} + \lambda_2 c_1 c_2 c_4 e^{-\theta_2}\right)\big)\big] \\
& \times \big[c_1 c_2\big({\lambda_2}^2 c_1 c_2 c_3 c_4 + \lambda_2\lambda_1 {c_2}^2 {c_3}^2 e^{2\theta_2 - 2\theta_1} + \lambda_2\lambda_1 c_1^2 c_4^2 e^{-2\theta_2 + 2\theta_1} + {\lambda_1}^2 c_1 c_2 c_3 c_4\big)\big]^{-1}
\end{aligned}
$$

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References

[1] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[2] Gu C H, Hu H S and Zhou Z X 1999 *Darboux Transformation in Soliton Theory and its Geometric Applications* (Shanghai: Shanghai Scientific and Technical Publishers)
[3] Levi D 1988 *Inverse Problem* **4** 165
[4] Gu C H and Hu H S 1986 *Lett. Math. Phys.* **11** 325
[5] Gu C H and Zhou Z X 1994 *Lett. Math. Phys.* **32** 1
[6] Estevez P G 1999 *J. Math. Phys.* **40** 1406
[7] Neugebauer G and Meinel R 1984 *Phys. Lett.* A **100** 467
[8] Fan E G 2005 *J. Phys. A: Math. Gen.* **38** 1063
[9] Cao C W and Geng X G 1991 *J. Math. Phys.* **32** 2323
[10] Yan Z Y and Zhang H Q 2002 *Chaos Solitons Fractals* **13** 1439
[11] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095